935 STATISTICAL MECHANICS - Homework #3. Due: TH, 3/2/2006.

1. This problem is about the method of Steepest Descents. The equivalence of ensembles can be shown using steepest descents. Use the method of steepest descents (you may consult your favorite math methods textbooks) to find the behavior of

$$J_\nu(z) = \frac{1}{2\pi}\int_0^{2\pi} e^{i\nu\theta + iz\cos\theta} d\theta \ , \tag{1}$$

for large z.

2. This problem is about the Isobaric Ensemble. Consider a system connected to a heat bath by rubber walls. The volume V therefore fluctuates, although pressure P and particle number N have definite values.

   (a) By analogy with the method used when discussing the canonical ensemble, argue that the probability $\mathcal{P}$ of finding oneself in a small region of phase space with system volume V is

$$\mathcal{P} d\Gamma = e^{-\alpha V - \beta E} d\Gamma \ . \tag{2}$$

   (b) Write down the complete and correct equation for the partition function Q_{iso}. Also write down expressions for the average value of E, the average value of V, and write down the relationship between Q_{can} and Q_{iso}.

   (c) Identify $\log Q_{iso}$ with an appropriate thermodynamic potential, multiplying by whatever factors are necessary to make the identification complete. Explicitly differentiating by β and α, show that your expression satisfies appropriate thermodynamic identities, and in carrying out the second part, find the identity of α, just as we found $\beta = 1/kT$ in class.

   (d) Calculate Q_{iso} for an ideal gas. From it, calculate $\mu(P, T)$ and $V(P, T)$.

3. This problem is about the Equivalence of Canonical and Isobaric Ensembles.

   (a) Look up the inversion formula for the Laplace transform, and find an expression for the canonical partition function $Q_{can}(\beta, V, N)$ as an integral in the complex plane of the isobaric partition function $Q_{iso}(\beta, \alpha, N)$. You will be integrating over complex α.

   (b) Write

$$Q_{iso} = e^{-Nq(\alpha,\beta)} \ , \tag{3}$$

   where q is a function you should determine using the results of the previous problem. Use the identity $\Phi = \mu N$ to eliminate Φ.

   (c) Insert your expression for Q_{iso} into the integral. Expand the argument of the exponent to quadratic order in the integration variable. Identify all of the coefficients in this expansion using thermodynamic relations. Now evaluate this integral using steepest descents in the limit $N \to \infty$. Use the result to show the equivalence of the isobaric and canonical ensembles in the thermodynamic limit, $N \to \infty$.