935 STATISTICAL MECHANICS - Final-Exam. Due: TH, 5/11/2006.

1. This problem is about Boltzmann's Constant. We have used Boltzmann's constant throughout the course. But how would one measure it? To answer this question you are asked to find **two separate** experimental procedures by which to measure the constant. In the discussion of each procedure you should include:

   (a) the physical principle which allows Boltzmann's constant to be measured in this way; that is, the equations upon which the measurement is based, and, briefly, where they come from;

   (b) in as specific a way as possible, the apparatus you would use to carry out the experiment;

   (c) your best quantitative estimate of the accuracy of the experiment.

   You may assume that physical constants unrelated to k_B are known – $\hbar$, c and so forth – but you cannot assume that you know constants such that dividing one by the other gives k_B. The purpose of the question is to have you design two experiments. If the experiment you design fills a large and important gap on the road to finding k_B, you may in general take all other needed constants to be known.

   If you make use of books, journals or the web, you must give credit to your sources.

   Please remember the importance of clarity of thought and presentation.

2. This problem is about Antiferroelectric Crystals. The energy levels of an antiferroelectric crystal of N molecules in no electric field ($\mathcal{E} = 0$) are given by

$$E_{N,\ell} \;=\; -\frac{1}{2}N\epsilon \;+\; \ell\epsilon \;, \tag{1}$$

   and each has degeneracy

$$\omega_{N,0} \;=\; 2 \;\;,\;\; \omega_{N,1} \;=\; 2N \;\;,\;\; \omega_{N,2} \;=\; N(N-1) \;\;,\;\; \omega_{N,\ell} \;=\; 2\frac{N!}{\ell!(N-\ell)!} \tag{2}$$

   where $\ell = 0, 1, 2, \ldots, N$. Here $E_{N,\ell}$ gives the energy level of the whole crystal, while ϵ is a microscopic energy scale. When an electric field is turned on, the degenerate states are split into two groups, and the change in the energies of the states is given by

$$\Delta E_{N,\ell} \;=\; \pm\mu\ell\mathcal{E} \;, \tag{3}$$

   where μ is some constant.

   (a) Sketch the ground-state energy U as a function of the electric field.

   (b) Calculate the partition function $Q_N(T, \mathcal{E})$ and from it find the free energy per molecule in the limit of large N.

   (c) Find the energy per molecule $U(T, \mathcal{E})$ and show that one recovers part (a) in the limit $T \to 0$.

(d) Find the entropy per molecule for $T \to \infty$ and for $T \to 0$, paying special attention to the field values $\pm\mathcal{E}_0 = \pm\epsilon/\mu$. Graph S for $T = 0$, and for T near zero. Discuss in relation to the third law.

(e) Find the electric polarization from the free energy. Find the polarization in zero electric field for a finite system (N finite) and compare with the limit $\mathcal{E} \to 0$ for a system in which N has already been taken infinite.

3. This problem is about the <u>one-dimensional spin-one Ising Model in zero field</u>, described by the Hamiltonian

$$H_N\{\sigma_i\} = -J\sum_i \sigma_i\sigma_{i+1} \qquad ; \qquad \sigma_i = -1, 0, +1 \; . \tag{4}$$

(a) Write down the transfer matrix $\mathbf{P}$ for this model.

(b) Show that the free energy of this model is given by

$$\frac{1}{N}F(T) = -kT\ln\{\frac{1}{2}[(1 + 2\cosh\beta J) + \sqrt{8 + (2\cosh\beta J - 1)^2}]\} \; . \tag{5}$$

Examine the limiting behavior of this quantity in the limits $T \to 0$ and $T \to \infty$.