944 QUANTUM MECHANICS - Homework #1. Due: Friday, September 21, 2007.

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1. This problem is about the orbital angular momentum operator, $\hat{L}_i = \epsilon_{ijk}\hat{X}_j\hat{P}_k$.

   (a) Using canonical commutation relations between components of $\hat{X}_i$ and $\hat{P}_i$, show that
   $$\left[\,\hat{X}_i, \hat{L}_j\,\right] \;=\; i\hbar\epsilon_{ijk}\hat{X}_k \quad \text{and} \quad \left[\,\hat{P}_i, \hat{L}_j\,\right] \;=\; i\hbar\epsilon_{ijk}\hat{P}_k\ . \tag{1}$$

   (b) Show that
   $$\left[\,\hat{L}_i, \hat{L}_j\,\right] \;=\; i\hbar\epsilon_{ijk}\hat{L}_k \quad \text{and} \quad \epsilon_{ijk}\hat{L}_j\hat{L}_k \;=\; i\hbar\hat{L}_i\ . \tag{2}$$

   (c) Define $\hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y$ and show that
   $$\left[\,\hat{L}_z, \hat{L}_\pm\,\right] \;=\; \pm\hbar\hat{L}_\pm \quad \text{and} \quad \left[\,\hat{L}_+, \hat{L}_-\,\right] \;=\; 2\hbar\hat{L}_z\ . \tag{3}$$

   (d) Show that in the spherical coordinate basis
   $$\begin{aligned}\hat{L}_z\ \Psi(r,\theta,\phi) &= -i\hbar\frac{\partial}{\partial\phi}\ \Psi(r,\theta,\phi)\ ,\\ \hat{L}_\pm\ \Psi(r,\theta,\phi) &= -i\hbar\ e^{\pm i\phi}\left(\pm i\frac{\partial}{\partial\theta} \;-\; \frac{1}{\tan\theta}\frac{\partial}{\partial\phi}\right)\ \Psi(r,\theta,\phi)\ .\end{aligned} \tag{4}$$

   (e) Compute $\hat{L}_i\hat{L}_i\ \Psi(r,\theta,\phi)$ in the same basis.
   (Hint: use $\hat{L}_i\hat{L}_i = \hat{L}_z^2 + \frac{1}{2}\hat{L}_+\hat{L}_- + \frac{1}{2}\hat{L}_-\hat{L}_+$.)

2. This problem is about rotations in three dimensions. For a rotation by angle α around an axis of a unit vector $\vec{n}$, the rotation matrix is
   $$R_{ij}(\alpha,\vec{n}) \;=\; \delta_{ij}\cos\alpha \;-\; \epsilon_{ijk}n_k\sin\alpha \;+\; n_in_j(1-\cos\alpha)\ . \tag{5}$$
   A product of two rotations $R(\alpha',\vec{n}')\cdot R(\alpha'',\vec{n}'')$ is itself a rotation by some angle α around some axis $\vec{n}$. To determine α and $\vec{n}$, we demand
   $$R_{ij}(\alpha,\vec{n}) \;=\; R_{ik}(\alpha',\vec{n}')\cdot R_{jk}(\alpha'',\vec{n}'')\ , \tag{6}$$
   and then substitute eq. (5) and solve for α and $\vec{n}$.

   (a) Show that to second order in α' and α'',
   $$\alpha\vec{n} \;\simeq\; \alpha'\vec{n}' \;+\; \alpha''\vec{n}'' \;+\; \frac{1}{2}\alpha'\alpha''\vec{n}'\times\vec{n}''\ . \tag{7}$$

   (b) Prove $trR \equiv R_{ii} = 1 + 2\cos\alpha$ and $\epsilon_{ijk}R_{jk} = -2n_i\sin\alpha$ and use these formulas to derive exact expressions for $\cos\alpha$ and $\vec{n}\sin\alpha$ in terms of α', α'', $\vec{n}'$ and $\vec{n}''$.

3. This problem is about Pauli matrices and the relation between abstract rotations in spin space and rotations in three dimensions.

(a) Show that for any two vectors $\vec{a}$ and $\vec{b}$,

$$(\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma}) = (\vec{a}\cdot\vec{b})\mathbf{1} \; + \; i(\vec{a}\times\vec{b})\cdot\vec{\sigma}. \tag{8}$$

(b) Show that for any angle α and for any unit vector $\vec{n}$,

$$U(\alpha,\vec{n}) \equiv \exp(-i\frac{\alpha}{2}\vec{n}\cdot\vec{\sigma}) = \cos(\frac{\alpha}{2})\mathbf{1} - i\sin(\frac{\alpha}{2})\;\vec{n}\cdot\vec{\sigma}\;. \tag{9}$$

(c) Solve for α and $\vec{n}$ such that

$$U(\alpha,\vec{n}) \;\; = \;\; U(\alpha',\vec{n}')\;U(\alpha'',\vec{n}'')\;. \tag{10}$$

(d) The commutation relations between generators of a Lie algebra completely determine the products of their exponentials. Consider a vector of 2×2 hermitian matrices, $\vec{S}\equiv\frac{\hbar}{2}\vec{\sigma}$; commutation relations between its compontents are exactly the same as between the components $\hat{J}_i$ of the angular momentum operator. Since the definition of the U matrices amounts to $U(\alpha,\vec{n}) = \exp(\alpha\vec{n}\cdot\vec{S}/i\hbar)$, which is precisely analogous to $\hat{R}(\alpha,\vec{n}) = \exp(\alpha\vec{n}\cdot\vec{\hat{J}}/i\hbar)$, we expect the product rule for the U matrices to be exactly the same as for the unitary rotation operators $\hat{R}$, which in turn sould be indentical to the product rule for the R_{ij} rotation matrices given in eq. (6).

Verify that a solution of eq. (10) is automatically a solution of eq. (6).

Is the converse also true?