Classical Field Theory

Lagrangian Mechanics

Action

\[ S = \int dt \, L (q_i(t), \dot{q}_i(t)) \]

\[ \frac{\delta S}{\delta \dot{q}_i(t)} = 0 \Rightarrow \text{E.O.M.} \]

Explicitly,

\[ \delta S = \int dt \left( \frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right) \]

\[ = \int dt \left( \frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i + \text{other terms} = 0 \]

E.O.M.:

\[ \frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = 0 \]
Example: S.H.O. 1-d

\[ L = \frac{1}{2} x^2 - \frac{w^2}{2} x^2 \]

\[ \frac{\partial L}{\partial x} = \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = -w^2 x - \ddot{x} = 0 \]

\[ \Rightarrow \ddot{x} = -w^2 x \]

Lagrangian Field Theory

Scalar field \( \Phi = \Phi(x^\mu) \) is continuous family of D.O.F.

\( \{ \) Can think of as limit of countable D.O.F. (as in a space-time lattice) \( \} \)

Fundamental quantity is \( S \): the action

\[ S = \int L dt = \int Y(\Phi, \partial_x \Phi) d^4x \]
$2$ is Lagrange density

"Principle of least action."

When system evolves from one configuration to another between $t_1$ and $t_2$ it does so along a path for which $S$ is an extremum (usually minimum).

\[ 0 = \frac{dS}{dt} \]

\[ = \int d^4x \left( \frac{\partial L}{\partial \dot{q}} \delta q + \frac{\partial L}{\partial \dot{q} \partial q} \delta q \right) \]

\[ = \int d^4x \left( \frac{\partial L}{\partial \dot{q}} \delta \dot{q} - \frac{\partial L}{\partial q} \frac{\partial \dot{q}}{\partial \dot{q}} (2) \delta q \right) \]

Surface integral over S.T. boundary
A) $\delta \Omega$ is 0 at temporal boundaries as have initial and final field configurations.

B) Restrict ourselves to $\delta \Omega$ that vanish on spatial boundary

\[ \text{Surface term vanishes} \]

As $\delta \Omega$ is arbitrary,

\[ \partial \mu \left( \frac{\partial \Omega}{\partial (\partial \mu \Omega)} \right) - \frac{\partial \Omega}{\partial \Omega} = 0 \]

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(1 such equation for each field.)

Example: \( u - c \) field

\[ \chi = \frac{1}{2} \partial \mu \Omega \partial \nu \chi - \frac{1}{2} m^2 \chi^2 \]

\[ \partial \mu \frac{\partial \Omega}{\partial (\partial \mu \Omega)} = \partial \mu \chi \]

\[ \frac{\partial \Omega}{\partial \Omega} = -m^2 \chi \]
E.O.M. KLEIN-GORDON EQUATION

$$\Box \phi + m^2 \phi = 0$$
on-\n
$$(\partial^2 + m^2) \phi = 0$$

(relativistic version of S.M.O. equation)

Hamiltonian Mechanics

$$p_i = \frac{D}{Dq^i} \quad \text{Eq. of, } p_i \delta_{ij}$$

$$H = \sum p_i \dot{q}^i - L$$

Hamiltonian Field Theory

Lagrangian formulation well suited to relativity, as is manifestly Lorentz invariant.

Hamiltonian formulation is more intuitive as related to non-relativistic C.M.
But \[ H \rightarrow E' \neq E \] 

Boosts

Lost manifest Lorentz invariance!

Nevertheless, we will follow the Hamiltonian method.

By analogy w/ mechanics, assume

\[ \Pi(x^i) = \frac{\partial H}{\partial \dot{\phi}(x^i)} \]

\[ H = \int d^3x \ H = \int d^3x \left[ \Pi(x^i) \dot{\phi}(x^i) - L \right] \]

\[ H \text{ is Hamiltonian density} \]

Example: \[ V \text{-} \phi \text{ field} \]

\[ L = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \]

\[ = \frac{1}{2} \left( \partial_\mu \phi \right)^2 - \frac{1}{2} m^2 \phi^2 \]
\[
\mathcal{L} = \frac{1}{2} \dot{\phi}^2 - \frac{i}{2} (\nabla \cdot \phi)^2 - \frac{i}{2} m^2 \phi^2
\]

\[
\mathcal{L}(\phi) = \frac{1}{2} \dot{\phi}^2 - \frac{i}{2} \phi (\nabla \cdot \phi)
\]

\[
H = \int d^3x \left[ \frac{1}{2} \dot{\phi}^2 + \frac{i}{2} (\nabla \cdot \phi)^2 + \frac{i}{2} m^2 \phi^2 \right]
\]

Energy lost:
- moving in time
- moving in space
- existing at all

Noether's Theorem

Relation between symmetries + conservation laws

Consider infinitesimal transformations as \( \delta \phi \)
\( \varphi(x) \rightarrow \varphi'(x) = \varphi(x) + \alpha \Delta \varphi(x) \)

This transformation is symmetry if the E.E.O.M. are left invariant.

Hence, w.r.t. respect to \( \varnothing \),

\[ Y(x) \rightarrow Y(x) + \alpha \partial \mu J^\mu(x) \]

(surface term)

(vanishes at boundary of \( S,T \))

\( J^\mu(x) \) is the new field.

Compare this with direct variation of \( Y \)

\[ Y(x) \rightarrow Y(x) + \alpha \Delta Y \]
\[ d \Delta y = d \left( \frac{\partial^2}{\partial \phi} \Delta \phi + \frac{\partial}{\partial (\partial \phi)} \Delta (\phi, \phi) \right) \]

\[ = \frac{\partial^2}{\partial \phi} (\partial \Delta \phi) + \frac{\partial}{\partial (\partial \phi)} \partial (\partial \phi) \]

\[ = \frac{\partial}{\partial \phi} \left( \frac{\partial^2}{\partial (\partial \phi)} \delta \phi \right) + \delta \left[ \frac{\partial}{\partial \phi} - \partial \left( \frac{\partial}{\partial (\partial \phi)} \right) \right] \delta \phi \]

\[ t = 1 \text{ c.o.m.} \rightarrow 0 \]

Composing with previous \[ \equiv \]

\[ \partial \mu J^\mu(x) = \partial \mu \left( \frac{\partial^2}{\partial (\partial \phi)} \delta \phi \right) \]

\[ \textbf{Solution} \]

\[ J^\mu(x) = \frac{\partial^2}{\partial (\partial \phi)} \delta \phi - j^\mu(x) \]

where \[ \partial \mu j^\mu(x) = 0 \]

"current" \[ j^\mu(x) \] is conserved!

\[ \boxed{\text{D'utu:} \quad j^\mu = \frac{\partial^2}{\partial (\partial \phi)} \delta \phi - J^\mu} \]
For each continuous symmetry of \( \mathbb{Z} \)

conservation law

(Noether's Thm)

(Clear generalization to more fields)

Note: Conservation law \( \Rightarrow \) conserved charge

\[
A = \int j^0(x) \, d^3x
\]

\[
\frac{dA}{dt} = \int \partial_0 j^0(x) \, d^3x
\]

\[
= \int \partial_\mu j^\mu(x) \, d^3x + \text{surface terms}
\]

\[
\Rightarrow \frac{dA}{dt} = 0
\]

(Not on as solid ground!)
Example:

\( \mathcal{L} = \frac{1}{2} (\partial_{\mu} \phi)^2 \)

Invariance: \( \phi \rightarrow \phi + \alpha \) \( \alpha \) constant

Recall:

\( j^\mu (x) = \partial_{\mu} \Delta \phi - \frac{1}{2} \overline{\phi} \gamma^\mu \gamma^5 (\partial_{\mu} \phi) \) 

(not relevant here)

\( j^\mu = \overline{\phi} \gamma^\mu \phi \quad \partial \cdot j = \Box \phi = 0 \) 

\( (K-6 \text{ equation}) \)

\( \mathcal{L} = \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - m^2 |\phi|^2 \)

\( = \partial_{\mu} \phi \partial^{\mu} \phi^* - m^2 \phi \phi^* \)

\( \phi \) is complex valued field

\( K-6 \text{ equation?} \)

treat \( \phi, \phi^* \) as independent fields.
\[ \phi : \quad \partial_{\mu} (\partial^{\mu} \phi^a) + m^2 \phi^a = 0 \Rightarrow (D + m^2) \phi^a = 0 \]

\[ \phi^a : \quad (D + m^2) \phi = 0 \]

Invariance of 2:

\[ \phi \rightarrow e^{i \alpha} \phi \]

"phase transformation"

Interpretation:

\[ \phi \rightarrow \phi + i \alpha \phi + \alpha (\partial^2 \phi) \]

\[ \partial \Delta \phi = i \partial \phi \]

\[ \partial \Delta \phi^a = -i \partial \phi^a \]

Conserved current?

\[ j^\mu(x) = \frac{\partial \Delta \phi}{\partial (\partial^\mu \phi)} + \frac{\partial \Delta \phi^a}{\partial (\partial^\mu \phi^a)} \]

\[ = \left[ (\partial^\mu \phi^a)(i \phi^a) + (\partial^\mu \phi)(-i \phi) \right] \]

\[ = \left[ (\partial^\mu \phi^a)(i \phi^a) + (\partial^\mu \phi)(-i \phi) \right] \]
Conserved?

\[ \rho_j = i \left[ (\partial \Psi^\dagger) \Psi + \partial \cdot \Psi \Psi \right] \\
- \partial_k \Psi^\dagger \partial^k \Psi - \Psi^\dagger (\partial \Psi) \right) \]

(use \( k-l \))

\[ = i \left[ -m^2 |\Psi|^2 + m^2 |\Psi|^2 \right] = 0 \]

Noether's Theorem also applies to

Space-time transformations:

Consider \( S-T \) translations:

\[ x^\mu \rightarrow x^\mu - a^\mu \]

\( (a^\mu \) is constant)\)

1: \[ \phi (x) \rightarrow \phi (x + a) = \phi (x) + a^\mu \partial_\mu \phi (x) + O(a^2) \]

2: \[ y \rightarrow y - a^\mu \partial_\mu y = y + a^\mu \partial_\mu (J^\mu - j^\mu) \]

4 separately conserved currents:

\[ T^\mu = \partial_\mu J^\nu - \partial_\nu J^\mu \]

\[ \partial (\partial_\nu \Psi) \]

\[ \partial (\partial_\nu \Psi) \]
\[ T^\mu_\nu : \text{Energy-Momentum Tensor} \]

Let's look more carefully at \( DS \) with respect to \( S-T \) translations:

\[ x^\mu \rightarrow x^\mu' = x^\mu + l x^\mu \]

\[ \delta S = \int d^{4}x' \psi (x', \partial x', x') - \int d^{4}x \psi (0, \partial x, x) \]

\[ d^{4}x' = J (x') d^{4}x \quad (J \text{ is Jacobian}) \]

\[ \partial x'^\mu = \delta x^\mu + \partial x^\mu \delta x^\mu \]

\[ J (x') \cdot \text{det} \left( \frac{\partial x'^\mu}{\partial x^\mu} \right) = 1 + \partial x^\mu \left( \delta x^\mu \right) \]

\[ \delta S = \int d^{4}x \left( \delta x^2 + \partial x^\mu \delta x^\mu \right) \]

\( (\delta x^2 \equiv \partial x^2 \delta x + \partial \delta x \partial x + \frac{\delta x^2}{\delta x}) \)
So \( ds^2 = \int d^4x \left( \frac{\partial \Phi}{\partial \Phi} + \frac{\partial \Phi}{\partial \Phi} \right) \)

Source of extra piece!

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**Conservation Laws**

\[
T^{00} = \frac{\partial}{\partial \Phi} \Phi - 2
\]

\[
= \Phi(x) \, \frac{\partial}{\partial \Phi} \Phi(x) - 2 = \mathcal{H}
\]

So \( \mathcal{H} = \int d^3x \, T^{00} \)

\( \mathcal{H} \) is generator of time translations

\[
T^{0i} = \frac{\partial}{\partial \Phi} \frac{\partial \Phi}{\partial \Phi}
\]

So \( \pi^i = \int d^3x \, T^{0i} = -\int d^3x \, \Phi \frac{\partial}{\partial \Phi} \Phi \)

Momentum (carried by field) generator of space translations