Basic object in Functional Quantization:

\[ Z[J] = \int D\phi \exp \left[ i \int d^4x \left( \frac{\partial}{\partial J(x)} \phi(x) \right)^2 + J(x) \phi(x) \right] \]

\[ \text{Source Term} \]

Note:

\[ \frac{d}{dJ(x)} \exp \left[ i \int d^4y J(y) \phi(y) \right] = i \phi(x) \exp \left[ i \int d^4y J(y) \phi(y) \right] \]

\[ \therefore \]

\[ C_{01} T \{ \phi(x_1) \phi(x_2) \} = \int D\phi \phi(x_1) \phi(x_2) \frac{\exp \left[ i \int d^4x \phi(x) \right]}{\int D\phi \exp \left[ i \int d^4x \phi(x) \right]} \]

\[ = \frac{1}{Z[0]} \left( \frac{\delta}{\delta J(x)} \right) \left( -i \frac{\delta}{\delta J(x)} \right) Z[J] \bigg|_{J=0} \]

In free theory \( Z[J] \) can be written in explicit form.

Note:

\[ \frac{1}{2} \partial^2 \phi \partial^2 \phi = -\frac{1}{2} \text{Lagrangian} \phi + \text{total} \]

\[ \text{deltachar} \]
\[ \int d^4x \left[ Z_0(\phi) + \psi \phi \right] = \int d^4x \left[ \frac{i}{2} \psi \left( \frac{D}{2} - m^2 + i\epsilon \right) \psi + \frac{\partial}{\partial x^i} \phi \right] \]

Feynman prescription
\[ \text{correction factor for} \]
\[ \text{path integral} \]

\[
20(S) \int D\psi \exp \left\{ -i \left[ \frac{1}{2} \psi \left( D + m^2 - i\epsilon \right) \psi - \psi \left( \frac{1}{2} \phi \right) \right] d^4x \right\}
\]

\[ \text{generally functional for free scalar field.} \]

\[ \text{Note:} \ \phi \text{ does not obey} \ K\ell \text{ equation.} \]

As we note \( S \): \[ \psi \rightarrow \psi + \psi_0 \]

\[ \int \left[ \frac{1}{2} \psi \left( D + m^2 - i\epsilon \right) \psi - \left( \frac{1}{2} \phi \right) \right] d^4x \rightarrow \]

\[ \int \left[ \frac{1}{2} \psi \left( D + m^2 - i\epsilon \right) \psi + \psi \left( D + m^2 - i\epsilon \right) \psi_0 + \frac{1}{2} \psi_0 \left( D + m^2 - i\epsilon \right) \psi_0 - \psi \left( \frac{1}{2} \phi \right) \psi_0 - \psi_0 \left( \frac{1}{2} \phi \right) \right] d^4x \]

\[ \left\{ \text{Used} \ \int_\mathbb{C} D\psi = \int D\phi \right\} \]
Now choose \( \psi_0 \) such that:

\[
(D + m^2 - \text{i} \epsilon) \psi_0(x) = J(x)
\]

Then

\[
Z_0[J] = \int D\psi \exp \left\{ -\frac{1}{2} \int \left[ \frac{1}{2} \psi (D + m^2 - \text{i} \epsilon) \psi - \frac{1}{2} \psi_0 J \right] d^4x \right\}
\]

But we know solution to 4!!

Recall that \( (D + m^2 - \text{i} \epsilon) D_x(x) = -\text{i} \delta^D(x) \)

So

\[
\psi_0(x) = -\text{i} \int D_x(x-y) J(y) d^4y
\]

Plugging in above we have:

\[
Z_0[J] = Z_0[0] \exp \left\{ -\frac{1}{2} \int J(x) D_x(x-y) J(y) d^4x d^4y \right\}
\]

Normalizing

Now we can compute the \( Z \)-pt. function.

\[
Z[J] \langle \psi(0) \psi(0) \rangle = \left. \frac{1}{Z_0[0]} \left( \frac{1}{i} \frac{d}{dJ(x)} \right) \left( \frac{1}{i} \frac{d}{dJ(x)} \right) Z_0[J] \right|_{J = 0}
\]

\[
= \left. -\frac{1}{i} \int d^4y D_x(x-y) J(y) d^4y \int d^4x J(x) D_x(x-x_0) \right|_{J = 0}
\]

\[
= \frac{Z[J]}{Z_0} \left|_{J = 0} \right.
\]
\[ \text{Note that } \langle \Omega | T S (0, 310) \rangle = \langle \Omega | 0 (10) \rangle \]
\[ = \frac{1}{2 \omega (u) j \omega ^2} \int e^{-i j \omega (u) T \gamma} \gamma_{1} \frac{\partial X}{\partial J_3} | \gamma_{1} = 0 \]

To evaluate high order fudge factors adopt complex notation:

\[ \langle \Omega | T S (0, 310) \rangle = \frac{1}{2 \omega (u) j \omega ^2} \int \frac{\partial X}{\partial J_3} | \gamma_{1} = 0 \]

\[ = \frac{1}{2 \omega (u) j \omega ^2} \left[ -J_3 \frac{\partial X}{\partial J_3} e^{-i j \omega (u) T \gamma} \right]_{\gamma_{1} = 0} \]

\[ = \frac{1}{2 \omega (u) j \omega ^2} \left[ -D_{24} + J_3 \frac{\partial X}{\partial J_3} e^{-i j \omega (u) T \gamma} \right]_{\gamma_{1} = 0} \]

\[ = \frac{1}{2 \omega (u) j \omega ^2} \left[ D_{24} J_3 \frac{\partial X}{\partial J_3} + D_{24} J_3 \frac{\partial X}{\partial J_3} e^{-i j \omega (u) T \gamma} \right]_{\gamma_{1} = 0} \]

\[ \{ \langle \Omega | T S (0, 0, 132) \rangle = 0 \} \]

\[ = D_{24} D_{12} + D_{24} D_{13} + D_{14} D_{23} \]
Renormalization

Our approach to QFT is based on perturbation theory. As we will see, beyond leading order there are divergences that must be dealt with in order to define QFT.

Recall:

\[
\phi^4 \text{ Theory}
\]

\[
\sim \lambda \int d^4 x \left( \frac{1}{(2\pi)^4} \frac{g^2}{g^2 - m^2} \right) \left( \int d^4 x \frac{1}{g^2} \right)
\]

Has 4 powers of $g$ in numerator, 2 powers in denominator.

\Rightarrow \text{ quadratic divergence}

Example of an \textit{ultra-violet} divergence

Another example of $\mathcal{O}(\xi^2)$:

\[
\sim \lambda^2 \int d^4 q_1 \ d^4 q_2 \ \frac{d(\xi - q_1 - p_1 - p_2)}{(2\pi)^4 (2\pi)^4 \ (g^2 - m^2) (g^2 - m^2)}
\]

\[
\sim \lambda^2 \int \frac{d^4 q}{(2\pi)^4} \ \frac{1}{(g^2 - m^2)[(p_1 + p_2 - g)^2 - m^2]} 
\]
\[ \left( \sum_{i=1}^{n} \frac{d_i^2}{s_i^2} \right) \text{ logarithmic divergence} \]

How do we find degree of divergence of a generic graph?

**Proprietary:** \( q^{-2} \)

**Vertices:** \( q^n \cdot 5 \cdot 5 \)

**Loops:** If independent events occur without interaction then place.

Consider degree with:

- \( n \) vertices
- \( E \) external lines
- \( I \) internal lines
- \( L \) loops

**Supersaid Degree of Divergence:** \( D \)

\[ D = dL - 2I \]

Check:

- \( \bullet \) \( 27 \quad D = 4 \cdot 1 - 2 \cdot 1 = 2 \)
- \( \bigcirc \) \( 27 \quad D = 4 \cdot 1 - 2 \cdot 2 = 0 \)
Now would like to express $D$ in terms of $E$ and $n$.
(Must eliminate $I$ and $L$)

There are $I$ internal momenta and momentum conservation at each vertex + overall momentum conservation:

\[ \# \text{ of internal momenta} = L = I - (n-1) \]  \hspace{1cm} (1)

In 2D theory, there are $4n$ legs
(4 at each vertex)

\[ 4n = E + 2I \]  \hspace{1cm} (2)

So

\[ D = dL - 2I \]

\[ = d(I - n + 1) - 2I \]  \hspace{1cm} (3)

\[ = d(2n - E - n + 1) - (4n - E) \]  \hspace{1cm} (4)

\[ D = d - (\frac{d}{2} - 1)E + n(d - 4) \]

For $d = 4$ we have

\[ D = 4 - E \]

(checked):

\[ d = 4 \]

\[ D = 4 - 2 = 2 \]

\[ \checkmark \]

\[ d = 4 - 4 = 0 \]
This indicates that all divergs in \( E \geq 4 \)
will converge!!

As \( D \) depends on \( E \) only in \( d=4 \), there is only small number of divergent graphs and we hope that these effects can be eliminated by an infinite \underline{Renormalization} of various physical quantities.

If this is true, theory is said to be:

\underline{Renormalizable}

Aside: Which \( \phi^4 \) theory?

\[
\begin{align*}
\mu & = \beta + \gamma T \\
D & = d - (\frac{d}{2} - 1) E + n \left[ \frac{\Gamma (0 - 2) - d}{\mu} \right]
\end{align*}
\]

For \( d=4 \) we have

\[
D = 4 - E + n (4 - 4)
\]

\( 2 \phi^6 \Rightarrow \) non renormalizable \( (D = 4 - E + 2n) \)

\( 2 \phi^3 \Rightarrow \) "super" renormalizable \( (D = 4 - (5 - n)) \)
Note:
In $\mathcal{N}=4$ theory all graphs with $k=6$ should be convergent. Here we consider:

Hidden 2- and 4-point divergences!!

Here $D$ is "super-unit" degree of divergence.

Aside on dimensional analysis

Consider the $d$-dimensional action:

$$S = \int d^{d-2} \Phi$$

dimensions: $d-1$

$[L] = L^{d-1} = \Lambda^d$

$[\Phi] = [L]^{d-1} = \Lambda^{d-1}$

$[\phi] = \left[ L^{d-1} \right] = \Lambda^{d-1}$

$\phi_0$ is $d+2$ spin $\Phi$ field, so $[\phi] = [\Phi] = \Lambda^{d-1}$
Now consider instead: $2\Phi^r$

\[ J + [2] = \ell R = \Lambda^r \]

\[-5 + r(1 - d_n) = -d\]

\[\therefore \quad \delta = d + r - \frac{rd}{2}\]

\[d \Phi^4 : \quad [2] = \Lambda^{d-2}\]

\[d \Phi^3 : \quad [2] = \Lambda^{d-2} d_n\]

\[d \Phi^2 : \quad [2] = \Lambda^{d-2} \ell\]

Recall:

\[D = d - (\frac{d}{2} - 1) \ell^2 + n \left[ \frac{r}{2} (d-2) - d \right]\]

(Simplified above (obviously in $\mathbb{H}^2$ theory)

My nil \(\otimes\):

\[D = d - (\frac{d}{2} - 1) \ell^2 - n \delta\]

Hence remember all they must have $\delta \geq 0$