Path Integrals in Quantum Mechanics

- Useful to see alternate derivation of Feynman Rules
- Close analogy between Q.F.T. and STAT MECH.

Consider non-relativistic Q.M. in one dimension:

\[ H = \frac{p^2}{2m} + V(x) \]

Suppose we wish to describe a particle travelling from position \( x_a \) to position \( x_b \) in time \( T \).

Amplitude for this process is

\[ U(x_a, x_b; T) = \langle x_b | e^{-iHT/\hbar} | x_a \rangle \]

(in "canonical" Hamiltonian formalism)

Let's motivate a different mathematical expression for this amplitude.

\{ Superposition principle: amplitude is coherent sum of all ways in which process can take place. \}
In our example, consider the amplitude for each particular special path that you take as a pure phase. One expects:

\[ U(x_a, x_b; T) \sim \sum \text{phase} \int \text{phase} \]

\[ \sum \text{phase} \int \text{phase} \]

Indicates 1 path for every function \( x(t) \) that begins at \( x_a \) and ends at \( x_b \).

Aside on Functional

Functional: A function that maps functions to numbers.

E.g.: \( F[x(t)] \) is a functional.

Can be integrated over a set of functions \( x(t) \).

We will see more properties of functionals as we proceed.

What do we use for "phase" in \( U \)?

In classical limit there should be 1 path:

The classical path.
One possibility is that "stationary phase approach" picks out classical paths.

Recall: \[ T(k) = \int d\omega \, g(\omega) e^{i k T(\omega)} \]

S.P.A.: \[ T(k) \] depends only on critical points of \( f(\omega) \):

i.e. when \( f'(\omega)|_{\omega = \omega_c} = 0 \)

Hence would expect classical path \( x_0(\omega) \) to be given by:

\[
\frac{\delta}{\delta x_0(\omega)} \left( \phi_c(\omega) \right) \bigg|_{\omega = \omega_c} = 0
\]

However, we also know that

\[
\frac{\delta}{\delta x_0(\omega)} \left( \mathcal{S}[x_0(\omega)] \right) \bigg|_{\omega = \omega_c} = 0
\]

action

\[ S = \int d\tau + L \]

\[ \phi_c = \frac{S}{\hbar} \]

satisfies with \( L \)

\[
\{ \text{s.p.a. valid when } S \gg \hbar \}
\]
\( 5 \)
Define the path integral measure by:

\[
\int D\phi(x) = \prod_{i=1}^{N-1} \frac{1}{C(\phi)} \frac{d\phi_1}{C(\phi)} \frac{d\phi_2}{C(\phi)} \cdots \frac{d\phi_{N-1}}{C(\phi)} = \frac{1}{C(\phi)} \prod_{i=1}^{N-1} \int d\phi_i
\]

where for the definition will become clear below.

Now we would like to show equivalence of "canonical" and "path integral" approaches for the particular potential problem by showing that both amplitudes satisfy the same differential equation, with some initial condition.

Consider the addition of the very first term slice in discretized sum over paths:

\[
U(x',x;T) = \int \frac{d\phi}{C(\phi)} \exp \left[ \frac{i}{\hbar} \sum \phi, \overline{\phi} \right] U(x,\phi;\overline{\phi},0) \exp \left[ \frac{i}{\hbar} \int \phi(x',\phi) d\tau \right]
\]

Now consider limit \( E \to 0 \) oscillates rapidly!! constrains \( x' \) to be near \( x \) !

\( \Rightarrow 0 \) and...
Express in terms of \( \delta \in (x' - x_0) \):

\[
\begin{align*}
V(x_x + x') &= V(x_x + J_{\gamma_1}) \leq V(x_x) \\
U(x_x, x'; T - \epsilon) &= U(x_x, x_0 + \gamma_1; T - \epsilon) \leq \left[ 1 + \delta \frac{\partial}{\partial x_0} + \frac{\delta^2}{2!} \frac{\partial^2}{\partial x_0^2} + \ldots \right] U(x_x, x_0; T - \epsilon)
\end{align*}
\]

\[
U(x_x, x'_0; T) = \int_{-\infty}^{\infty} dx' \exp \left( \frac{i}{\hbar} \frac{m}{2} (x_x - x'_0)^2 \right) \left[ 1 - i \epsilon V(x_x) \right]
\]

\[
\times \left[ 1 + (x'_0 - x_0)^2 + \frac{i}{\hbar} \left( x'_0 - x_0 \right)^2 \frac{\partial^2}{\partial x_0^2} + \ldots \right] U(x_x, x_0; T - \epsilon)
\]

We can now do the "Gaussian" integral over \( x' \)!!

\[
\left\{ \begin{array}{l}
\int dy e^{-by^2} = \sqrt{\frac{\pi}{b}} \\
\int dy y e^{-by^2} = 0 \\
\int dy y^2 e^{-by^2} = \frac{1}{2b} \sqrt{\frac{\pi}{b}}
\end{array} \right.
\]

\[
\begin{align*}
U(x_x, x'_0; T) &= \left( \frac{1}{\sqrt{2\pi i \hbar}} \right) \left[ 1 - i \epsilon V(x_x) + i \epsilon \delta \frac{\partial}{\partial x_0} + o(\epsilon^2) \right] \\
&\times U(x_x, x_0; T - \epsilon)
\end{align*}
\]

As \( \epsilon \to 0 \) makes sense or \( r \) if \( (11) = 1 \)

\[
\begin{align*}
\epsilon(\epsilon) &= \sqrt{\frac{2\pi i \hbar}{-i m}}
\end{align*}
\]
\[ \text{Conclusion: } \quad H = U \wedge (x_1, x_2, x_3, \ldots) \]

\[ \text{Schönfinkel Lemma: } \quad \neg \exists \alpha \quad \neg \exists \beta \quad \neg \exists \gamma \quad \neg \exists \delta \]

\[ \left[ \neg \exists \alpha \quad \neg \exists \beta \quad \neg \exists \gamma \quad \neg \exists \delta \right] \]

\[ \left[ \neg \exists \alpha \quad \neg \exists \beta \quad \neg \exists \gamma \quad \neg \exists \delta \right] \text{ (I)} \]

\[ \neg \exists \alpha \quad \neg \exists \beta \quad \neg \exists \gamma \quad \neg \exists \delta \]

\[ \text{Now we can form a term } u(x_0, x_1, x_2, \ldots) \]