I. THE CROSS SECTION

1. The total cross-section, $\sigma$, is related to the probability that scattering occurs, and, as shown in class, is given by the formula

$$\sigma(p) = \int d\Omega \left| f_p(\Omega) \right|^2,$$

(1)

where $f_p$ is the scattering amplitude. As the cross-section carries units of distance-squared, one can visualize it as the effective cross-sectional area of the target particles "seen" by the beam particles. A more precise definition is

$$\sigma = \frac{\text{number of reactions per unit time}}{\text{beam particles per unit time} \times \text{scattering centers per unit area}}.$$

(2)

(a) Use eq. 2 to argue that the differential cross-section can be written as

$$\frac{d\sigma}{d\Omega} = r^2 \frac{|J_{\text{scattered}}|}{|J_{\text{incident}}|},$$

(3)

where $r$ is the radial coordinate and $J$ is the probability current.

(b) Use the asymptotic form of the total wavefunction, $\psi^+$, written in terms of an incoming plane wave and an outgoing spherical wave, together with eq. 3, to recover eq. 1.

II. THE BORN APPROXIMATION

2. For a central potential, $V(r)$, the Born approximation for the scattering amplitude may be written as

$$f_q(\Omega) = -\frac{m}{2\pi} \int d^3r' V(r') e^{-i\vec{q} \cdot \vec{p}'}$$

(4)

where $\vec{q} = \vec{p}' - \vec{p}$ is the "momentum transfer", and $\vec{p}$ and $\vec{p}'$ are the momenta of a particle (e.g. an $\alpha$-particle) before and after scattering, respectively. This particle scatters off a target (e.g. gold foil) at rest in the lab frame.

(a) Draw a picture of the scattering process described above and label all momenta and the scattering angle, $\theta$.

(b) Using energy conservation, find the relation between the magnitude of $\vec{q}$, the magnitude of $\vec{p}$, and the scattering angle $\theta$.

(c) Evaluate the angular part of the integral in eq. 4.
(d) Now assume a potential of the form

\[ V(r) = V_0 \frac{e^{-\alpha r}}{\alpha r} . \]  \hspace{1cm} (5)

Here \(1/\alpha\) may be viewed as the “range” of the potential. For instance, with \(\alpha = 0\), this is the Coulomb potential, mediated by photon exchange. Evaluate the scattering amplitude with arbitrary \(\alpha\).

(e) Finally, calculate the differential cross section, and discuss the \(\alpha \to 0\) limit. This is the Rutherford cross section. (HINT: note that the ratio \(V_0/\alpha\) must be finite as \(\alpha \to 0\).)

III. TOY MODEL OF THE NN SYSTEM

3. Here we will view the \(NN\) force as being due to the exchange of a spinless particle with mass equal to that of the pion,

\[ V(r) = -\beta \frac{e^{-m_{\pi}r}}{r} . \]  \hspace{1cm} (6)

We will choose \(\beta = 0.070\). Consider a single nucleon incident upon a single nucleon at rest in the lab frame. Denote the incident nucleon kinetic energy as \(T_{\text{lab}}\). Assume that the nucleons are interacting in the channel with total spin zero. (This problem will require numerical work.)

(a) Plot the s-wave radial wavefunction \(u_0(r)\) as a function of \(r\) for this potential, and on the same plot show the corresponding radial wavefunction \(u_0(r)\) with the potential turned off.

(b) Determine the s-wave scattering length.

(c) Determine the s-wave phase shift and plot it in degrees vs. \(T_{\text{lab}}\) between 0 and 100 MeV.

(d) Compare with experimental data available at: http://nn-online.sci.kun.nl/NN/NNonline.html, and comment on how good the agreement is.